Errata – chapter 2

Local volatility

2.4.2 Approximate expression for weakly local volatilities

Please find below a more explicit derivation of (2.35).

The price $P_0(t,S)$ of the K,T vanilla option in the model with time-dependent variance $u_0(t) = \sigma_0^2(t)$ is given by:

$$P_0(t,S) \;=\; P_{BS}(t,S,\widehat{\sigma}_0(t))$$

where P_{BS} is the Black-Scholes formula and implied volatility $\widehat{\sigma}_0(t)$ is given by:

$$\widehat{\sigma}_0(t)^2 \;=\; \frac{1}{T-t}\int_t^T u_0(\tau)\,d\tau$$

Consider now a perturbation of $u_0(t)$ by $\delta u(t,S)$:

$$u(t,S) \;=\; u_0(t) + \delta u(t,S)$$

resulting in a variation of the K,T vanilla option price by δP. From (2.30), at order one in $\delta u(t,S)$, δP is given by:

$$\begin{aligned}
\delta P \;&=\; E_{\sigma_0(t)}\left[\int_0^T e^{-rt}\frac{S_t^2}{2}\frac{d^2P_0}{dS^2}(t,S_t)\;\delta u(t,S_t)\,dt\right] \\
&=\; E_{\sigma_0(t)}\left[\int_0^T e^{-rt}\frac{S_t^2}{2}\frac{d^2P_{BS}}{dS^2}(t,S_t,\widehat{\sigma}_0(t))\;\delta u(t,S_t)\,dt\right] \qquad (2.1)
\end{aligned}$$

where $E_{\sigma_0(t)}$ denotes an expectation taken with respect to the density generated by the deterministic time-dependent volatility $\sigma_0(t)$. When $\delta u = 0$, P_0 is given by the Black-Scholes formula with implied volatility $\widehat{\sigma}_0 \;\equiv\; \widehat{\sigma}_0(0)$. We look for the perturbation $\delta\widehat{\sigma}$ of the implied volatility that generates the price variation δP.

Consider as base model the Black-Scholes model with constant implied volatility $\widehat{\sigma}_0$ and consider a perturbation $\delta\widehat{\sigma}$ of $\widehat{\sigma}_0$. Using again (2.30), at order one in $\delta\widehat{\sigma}$ the resulting price variation δP is given by:

$$\begin{aligned}
\delta P \;&=\; E_{\widehat{\sigma}_0}\left[\int_0^T e^{-rt}\frac{S_t^2}{2}\frac{d^2P_{BS}}{dS^2}(t,S_t,\widehat{\sigma}_0)\;\delta\left(\widehat{\sigma}^2\right)dt\right] \\
&=\; \delta\left(\widehat{\sigma}^2\right)E_{\widehat{\sigma}_0}\left[\int_0^T e^{-rt}\frac{S_t^2}{2}\frac{d^2P_{BS}}{dS^2}(t,S_t,\widehat{\sigma}_0)\,dt\right]
\end{aligned}$$

Now $e^{-rt}\frac{S_t^2}{2}\frac{d^2P_{BS}}{dS^2}(t,S_t,\widehat{\sigma}_0)$ is a martingale – see a proof in Appendix A of Chapter 5, page 181 – thus:

$$
\begin{aligned}
E_{\widehat{\sigma}_0}\left[e^{-rt}\frac{S_t^2}{2}\frac{d^2P_{BS}}{dS^2}(t,S_t,\widehat{\sigma}_0)\right] &= E_{\widehat{\sigma}_0}\left[e^{-rT}\frac{S_T^2}{2}\frac{d^2P}{dS^2}(T,S_T)\right] \\
&= E_{\sigma_0(t)}\left[e^{-rT}\frac{S_T^2}{2}\frac{d^2P}{dS^2}(T,S_T)\right] \qquad (2.2) \\
&= E_{\sigma_0(t)}\left[e^{-rt}\frac{S_t^2}{2}\frac{d^2P_{BS}}{dS^2}(t,S_t,\widehat{\sigma}_0(t))\right] \qquad (2.3)
\end{aligned}
$$

where (2.2) follows from the fact that the densities of S_T in (a) a Black-Scholes model with constant volatility $\widehat{\sigma}_0$, (b) in a Black-Scholes model with deterministic volatility $\sigma_0(t)$ are identical, and (2.3) again follows from the martingality of $e^{-rt}\frac{S_t^2}{2}\frac{d^2P_{BS}}{dS^2}(t,S_t,\widehat{\sigma}_0(t))$. We thus have:

$$
\delta P = \delta\left(\widehat{\sigma}^2\right)\ E_{\sigma_0(t)}\left[\int_0^T e^{-rt}\frac{S_t^2}{2}\frac{d^2P_{BS}}{dS^2}(t,S_t,\widehat{\sigma}_0(t))\,dt\right]
$$

Equating now this expression for δP with that in equation (2.1), to get $\delta\left(\widehat{\sigma}^2\right)$ at order one in δu, we get:

$$
\delta\left(\widehat{\sigma}^2\right) = \frac{E_{\sigma_0(t)}\left[\int_0^T e^{-rt}\frac{S_t^2}{2}\frac{d^2P_{BS}}{dS^2}(t,S_t,\widehat{\sigma}_0(t))\ \delta u(t,S_t)\,dt\right]}{E_{\sigma_0(t)}\left[\int_0^T e^{-rt}\frac{S_t^2}{2}\frac{d^2P_{BS}}{dS^2}(t,S_t,\widehat{\sigma}_0(t))\,dt\right]}
$$

which is the result in (2.35).

A.3 Using the UVM to price transaction costs

Page 93:

Since δS if or order $\sqrt{\delta t}$ …

should read:

Since δS is of order $\sqrt{\delta t}$ …