The local volatility model

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Intro – things heard on the street

- LV model used inconsistently: local vol surface is calibrated today; only to be recalibrated tomorrow.
  ⇨ violates model's assumption of fixed LV surface.
- Trading practice: don't use LV delta – instead compute "sticky-strike" delta: move S, keep implied vols unchanged, recalibrate local vol surface.
  - Rationale: so that vanilla options have BS delta.
- On a scale from dirty to downright ugly, where do we stand?
  - What is the carry P&L of an option position?
  - By the way, what's the delta of a vanilla option?

Outline

- Usable models?
- The carry P&L of the LV model
- The delta – the delta of a vanilla option
- Break-even levels for vols of implied vols / covariance of spot and implied volatilities
- Conclusion

⇨ Material of this presentation part of forthcoming book

Practically usable market models

- Define assets A_i to be used as hedging instruments – pricing function $P(t, A_1, \ldots, A_n)$
- First order contribution to P&L cancelled by delta hedging:

$$P\&L \;=\; -\left[P\left(t+\delta t, A+\delta A\right) - P\left(t, A\right)\right] + \Sigma \frac{dP}{dA_i}\left(\delta A_i - \mu_i A_i \delta t\right)$$

- First non-trivial contribution to P&L – order 1 in δt, 2 in δA:

$$P\&L \;=\; -\frac{dP}{dt}\delta t \;-\; \frac{1}{2}\Sigma\frac{d^2P}{dA_i dA_j}\delta A_i \delta A_j$$

- Condition for non-nonsensical P&L: there exists a positive breakeven covariance matrix $C_{ij}(t, A)$ such that:

$$\frac{dP}{dt} \;=\; \frac{1}{2}\Sigma\frac{d^2P}{dA_i dA_j}A_i A_j C_{ij}$$

$$P\&L \;=\; -\frac{1}{2}\Sigma A_i A_j \frac{d^2P}{dA_i dA_j}\left(\frac{\delta A_i}{A_i}\frac{\delta A_j}{A_j} - C_{ij}\delta t\right)$$

  - Trademark of a market model
  - Sign of P&L depends on mismatch between model & realized covariances

- Ideally we would like to set the C_{ij} – otherwise may need to use implied levels.

The local volatility model

- In LV model $P^{\mathrm{LV}}(t,S,\sigma)$. $\sigma(t,S)$ local volatility function.
- Use "local volatility delta" $\Delta^{\mathrm{LV}} = \left.\frac{dP^{\mathrm{LV}}}{dS}\right|_{\sigma}$. From pricing equation, P&L during δt:

$$P\&L^{LV} = -\frac{1}{2}S^2\frac{d^2P^{\mathrm{LV}}}{dS^2}\left(\left(\frac{\delta S}{S}\right)^2 - \sigma^2(t,S)\,\delta t\right)$$

- In LV model – with fixed LV function – implied vols functions of t, S:

$$\widehat{\sigma}_{KT}(t,S) \equiv \Sigma^{\mathrm{LV}}_{KT}(t,S,\sigma)$$

- $P\&L^{LV}$ actual P&L only if market implied vols move as prescribed by $\Sigma^{\mathrm{LV}}_{KT}(t,S,\sigma)$.

$$\Rightarrow \quad \Delta^{\mathrm{LV}} \text{ useless}$$

- Pricing function $P(t,S,\widehat{\sigma}_{KT})$. LV function σ calibrated on vanilla smile:

$$P(t,S,\widehat{\sigma}_{KT}) \equiv P^{\mathrm{LV}}(t,S,\sigma[t,S,\widehat{\sigma}_{KT}])$$

$$P^{\mathrm{LV}}(t,S,\sigma) = P\left(t,S,\Sigma^{\mathrm{LV}}_{KT}(t,S,\sigma)\right)$$

- Let's compute the carry P&L of an option position.

Carry P&L with the LV model – (black-box) pricing function $P\left(t, S, \widehat{\sigma}_{KT}\right)$

- Start with P&L of *naked* option position during δt:

$$P\&L \;=\; -\left[P(t+\delta t,\ S+\delta S,\ \widehat{\sigma}_{KT}+\delta\widehat{\sigma}_{KT}) - (1+r\delta t)P(t,S,\widehat{\sigma}_{KT})\right]$$

- Expanding at order one in δt and two in δS and $\delta\widehat{\sigma}_{KT}$:

$$\begin{aligned} P\&L \;=\; & rP\delta t \\ & -\frac{dP}{dt}\delta t \;-\; \frac{dP}{dS}\delta S \;-\; \frac{dP}{d\widehat{\sigma}_{KT}} \bullet \delta\widehat{\sigma}_{KT} \\ & -\left(\frac{1}{2}\frac{d^2P}{dS^2}\delta S^2 + \frac{d^2P}{dSd\widehat{\sigma}_{KT}} \bullet \delta\widehat{\sigma}_{KT}\delta S + \frac{1}{2}\frac{d^2P}{d\widehat{\sigma}_{KT}d\widehat{\sigma}_{K'T'}} \bullet \delta\widehat{\sigma}_{KT}\delta\widehat{\sigma}_{K'T'}\right) \end{aligned}$$

- Notation $\bullet$ stands for:

$$\frac{df}{d\widehat{\sigma}_{KT}} \bullet \delta\widehat{\sigma}_{KT} \;\equiv\; \iint dKdT\,\frac{\delta f}{\delta\widehat{\sigma}_{KT}}\;\delta\widehat{\sigma}_{KT} \;\equiv\; \Sigma_{ij}\frac{df}{d\widehat{\sigma}_{K_iT_j}}\;\delta\widehat{\sigma}_{K_iT_j}$$

- $\frac{dP}{dS}, \frac{dP}{dt}$ are computed keeping the $\widehat{\sigma}_{KT}$ fixed – the LV function is *not* fixed.
- Define sticky-strike delta Δ^{SS}:

$$\Delta^{SS} \;=\; \left.\frac{dP}{dS}\right|_{\widehat{\sigma}_{KT}}$$

Carry P&L with the LV model – 2

- Utilize that P is given by LV pricing equation:

$$P^{\text{LV}}(t,S,\sigma) \;=\; P\left(t,S,\widehat{\sigma}_{KT} = \Sigma^{\text{LV}}_{KT}(t,S,\sigma)\right)$$

- Express derivatives of P^{LV} in terms of derivatives of P:

$$\frac{dP^{\text{LV}}}{dt} \;=\; \frac{dP}{dt} + \frac{dP}{d\widehat{\sigma}_{KT}} \bullet \frac{d\Sigma^{\text{LV}}_{KT}}{dt}$$

$$\frac{dP^{\text{LV}}}{dS} \;=\; \frac{dP}{dS} + \frac{dP}{d\widehat{\sigma}_{KT}} \bullet \frac{d\Sigma^{\text{LV}}_{KT}}{dS}$$

$$\frac{d^2P^{\text{LV}}}{dS^2} \;=\; \left(\frac{d^2P}{dS^2} \;+\; 2\frac{d^2P}{dSd\widehat{\sigma}_{KT}} \bullet \frac{d\Sigma^{\text{LV}}_{KT}}{dS} \;+\; \frac{d^2P}{d\widehat{\sigma}_{KT}d\widehat{\sigma}_{K'T'}} \bullet \frac{d\Sigma^{\text{LV}}_{KT}}{dS}\frac{d\Sigma^{\text{LV}}_{K'T'}}{dS}\right) \;+\; \frac{dP}{d\widehat{\sigma}_{KT}} \bullet \frac{d^2\Sigma^{\text{LV}}_{KT}}{dS^2}$$

- Now insert these expressions in LV pricing equation:

$$\frac{dP^{\text{LV}}}{dt} \;+\; (r-q)S\frac{dP^{\text{LV}}}{dS} \;+\; \frac{1}{2}\sigma^2(t,S)\,S^2\frac{d^2P^{\text{LV}}}{dS^2} \;=\; rP^{\text{LV}}$$

- ... to generate relationship involving derivatives of P.

Carry P&L with the LV model – 3

- Yields:

$$\frac{dP}{dt} = rP - (r-q)S\frac{dP}{dS} - \frac{dP}{d\widehat{\sigma}_{KT}} \bullet \mu_{KT}$$
$$- \frac{1}{2}\sigma^2(t,S)\,S^2\left(\frac{d^2P}{dS^2} + 2\frac{d^2P}{dSd\widehat{\sigma}_{KT}} \bullet \frac{d\Sigma^{\text{LV}}_{KT}}{dS} + \frac{d^2P}{d\widehat{\sigma}_{KT}d\widehat{\sigma}_{K'T'}} \bullet \frac{d\Sigma^{\text{LV}}_{KT}}{dS}\frac{d\Sigma^{\text{LV}}_{K'T'}}{dS}\right)$$

with μ_{KT} given by:

$$\mu_{KT} = \frac{d\Sigma^{\text{LV}}_{KT}}{dt} + \frac{1}{2}\sigma^2(t,S)\,S^2\frac{d^2\Sigma^{\text{LV}}_{KT}}{dS^2} + (r-q)S\frac{d\Sigma^{\text{LV}}_{KT}}{dS}$$

- Now rewrite P&L of *naked* option position in slide 6:

$$P\&L = -\frac{dP}{dS}\left(\delta S - (r-q)S\delta t\right) - \frac{dP}{d\widehat{\sigma}_{KT}} \bullet \left(\delta\widehat{\sigma}_{KT} - \mu_{KT}\delta t\right)$$
$$+ \frac{1}{2}\sigma^2(t,S)\,S^2\left(\frac{d^2P}{dS^2} + 2\frac{d^2P}{dSd\widehat{\sigma}_{KT}} \bullet \frac{d\Sigma^{\text{LV}}_{KT}}{dS} + \frac{d^2P}{d\widehat{\sigma}_{KT}d\widehat{\sigma}_{K'T'}} \bullet \frac{d\Sigma^{\text{LV}}_{KT}}{dS}\frac{d\Sigma^{\text{LV}}_{K'T'}}{dS}\right)\delta t$$
$$- \left(\frac{1}{2}\frac{d^2P}{dS^2}\delta S^2 + \frac{d^2P}{dSd\widehat{\sigma}_{KT}} \bullet \delta\widehat{\sigma}_{KT}\delta S + \frac{1}{2}\frac{d^2P}{d\widehat{\sigma}_{KT}d\widehat{\sigma}_{K'T'}} \bullet \delta\widehat{\sigma}_{KT}\delta\widehat{\sigma}_{K'T'}\right)$$

Carry P&L with the LV model – 4

- Introduce implied (log-normal) vol of vol of $\widehat{\sigma}_{KT}$:

$$\nu_{KT} = \frac{1}{\Sigma_{KT}^{\text{LV}}} \frac{d\Sigma_{KT}^{\text{LV}}}{dS} S\sigma(t,S)$$

- Rewrite P&L as:

$$\begin{aligned}
P\&L = & -\frac{dP}{dS}\left(\delta S-(r-q)S\delta t\right) - \frac{dP}{d\widehat{\sigma}_{KT}} \bullet \left(\delta\widehat{\sigma}_{KT}-\mu_{KT}\delta t\right) \\
& -\frac{1}{2}S^2\frac{d^2P}{dS^2}\left[\frac{\delta S^2}{S^2}-\sigma^2(t,S)\,\delta t\right] \\
& -\frac{d^2P}{dSd\widehat{\sigma}_{KT}} \bullet S\widehat{\sigma}_{KT}\left[\frac{\delta S}{S}\frac{\delta\widehat{\sigma}_{KT}}{\widehat{\sigma}_{KT}}-\sigma(t,S)\,\nu_{KT}\delta t\right] \\
& -\frac{1}{2}\frac{d^2P}{d\widehat{\sigma}_{KT}d\widehat{\sigma}_{K'T'}} \bullet \widehat{\sigma}_{KT}\widehat{\sigma}_{K'T'}\left[\frac{\delta\widehat{\sigma}_{KT}}{\widehat{\sigma}_{KT}}\frac{\delta\widehat{\sigma}_{K'T'}}{\widehat{\sigma}_{K'T'}}-\nu_{KT}\nu_{K'T'}\delta t\right]
\end{aligned}$$

- Only uses market observables: $P(t,S,\widehat{\sigma}_{KT})$ – no LV function involved.
- P&L expression is that of market model.
  - Variance/covariance breakeven levels are well-defined + make up a positive covariance matrix.
  - Delta is sticky-strike delta $\frac{dP}{dS}$, vegas simple vegas.

Carry P&L with the LV model – 5

- $\widehat{\sigma}_{KT} \equiv$ implied vol plays no special role. Use instead price O_{KT}: $\mathcal{P}(t, S, O_{KT})$.

$$\begin{aligned} P(t,S,\widehat{\sigma}_{KT}) &= \mathcal{P}\big(t,S,O_{KT} = P^{\mathrm{BS}}_{KT}(t,S,\widehat{\sigma}_{KT})\big) \\ P^{\mathrm{LV}}(t,S,\sigma) &= \mathcal{P}\big(t,S,\Omega^{LV}_{KT}(t,S,\sigma)\big) \end{aligned}$$

  $\Omega^{LV}_{KT}(t,S,\sigma)$ price in LV model with LV function σ.

- Drift μ_{KT} simplifies:

$$\mu_{KT} = \frac{d\Omega^{\mathrm{LV}}_{KT}}{dt} + \frac{1}{2}\sigma^2(t,S)\,S^2\frac{d^2\Omega^{\mathrm{LV}}_{KT}}{dS^2} + (r-q)S\frac{d\Omega^{\mathrm{LV}}_{KT}}{dS} = r\Omega^{\mathrm{LV}}_{KT} = rO_{KT} \qquad \text{OK}$$

- P&L of naked option position – using now asset prices – cf slide 4:

$$\begin{aligned} P\&L = & -\frac{d\mathcal{P}}{dS}\left(\delta S - (r-q)S\delta t\right) - \frac{d\mathcal{P}}{dO_{KT}} \bullet \left(\delta O_{KT} - rO_{KT}\delta t\right) \\ & -\frac{1}{2}\frac{d^2\mathcal{P}}{dS^2}\left[\delta S^2 - \sigma^2(t,S)\,S^2\delta t\right] \\ & -\frac{d^2\mathcal{P}}{dSdO_{KT}} \bullet \left[\delta S\delta O_{KT} - \sigma^2(t,S)\,S^2\frac{d\Omega^{LV}_{KT}}{dS}\delta t\right] \\ & -\frac{1}{2}\frac{d^2\mathcal{P}}{dO_{KT}dO_{K'T'}} \bullet \left[\delta O_{KT}\delta O_{K'T'} - \sigma^2(t,S)\,S^2\frac{d\Omega^{LV}_{KT}}{dS}\frac{d\Omega^{LV}_{K'T'}}{dS}\delta t\right] \end{aligned}$$

Carry P&L with the LV model – conclusion

- P&L expression has typical form of market models – OK
- Hedging instruments all treated on equal footing.
- Implied break-even levels of cross-gammas are payoff-independent – are determined by market smile prevailing at time t.
  - spot/vol correl $= -100\%$
  - vol/vol correl $= 100\%$
  - vol of $\widehat{\sigma}_{KT}$ is $\nu_{KT} = \frac{1}{\Sigma_{KT}^{\mathrm{LV}}} \frac{d\Sigma_{KT}^{\mathrm{LV}}}{dS} S\sigma(t, S)$
- Hedge ratios are simply $\left.\frac{d\mathcal{P}}{dS}\right|_{O_{KT}}$ and $\left.\frac{d\mathcal{P}}{dO_{KT}}\right|_{S}$
- Delta of the local volatility model is $\Delta^{\mathrm{MM}} = \left.\frac{d\mathcal{P}}{dS}\right|_{O_{KT}}$.
- Delta of vanilla option is an irrelevant notion.
  - akin to asking model to generate a hedge ratio of one hedging instrument on another hedging instrument.
- Result seems $\approx$ natural, but see 2nd talk this afternoon for strange pathologies in local/stoch vol models.

So, what is the LV model?

- The LV model is a market model for the underlying and vanilla options
  ... that happens to have a 1-d Markov representation in terms of (t, S).
- This is a mathematical technicality – of which the LV function is a by-product – that facilitates pricing. Nothing fundamental.
- Daily recalibration of LV function is exactly how it has to be used.
- Consequences of 1-d Markov representation:
  - The break-even covariance matrix is of rank 1 – correls $= 100\%$.
  - No control on break-even levels of volatilities of implied volatilities. They are set by the configuration of S, $\widehat{\sigma}_{KT}$ and will vary unpredictably.
    - Like them, use model; don't like them, don't use model.
- This is how much we can get in a model with a 1-d Markov representation.

Consistency of sticky-strike and market-model deltas

- Use $S, O_{KT} \Rightarrow \mathcal{P}(t, S, O_{KT})$. Hedge ratios $\Delta^{\text{MM}} = \left.\frac{d\mathcal{P}}{dS}\right|_{O_{KT}}, \quad \left.\frac{d\mathcal{P}}{dO_{KT}}\right|_{S}$
- Use $S, \widehat{\sigma}_{KT} \Rightarrow P(t, S, \widehat{\sigma}_{KT})$. Hedge ratios $\Delta^{\text{SS}} = \left.\frac{dP}{dS}\right|_{\widehat{\sigma}_{KT}}, \quad \left.\frac{dP}{d\widehat{\sigma}_{KT}}\right|_{S}$
  - $\frac{dP}{d\widehat{\sigma}_{KT}}$ offset by trading *BS-delta-hedged* vanilla options
- Hedge portfolio is:

$$\Pi = \frac{d\mathcal{P}}{dS} S + \frac{d\mathcal{P}}{dO_{KT}} \bullet O_{KT}$$

- Rewrite in terms of delta-hedged vanillas:

$$\Pi = \left[\frac{d\mathcal{P}}{dS} + \frac{d\mathcal{P}}{dO_{KT}} \bullet \frac{dP^{\text{BS}}_{KT}}{dS}\right] S + \frac{d\mathcal{P}}{dO_{KT}} \bullet \left[O_{KT} - \frac{dP^{\text{BS}}_{KT}}{dS} S\right]$$

- Spot hedge ratio?
  - Move spot + move vanilla prices by their Black-Scholes deltas
    akin to: move vanilla prices keeping implied vols fixed $\Rightarrow$ sticky strike delta

$$\Delta^{\text{SS}} = \frac{d\mathcal{P}}{dS} + \frac{d\mathcal{P}}{dO_{KT}} \bullet \frac{dP^{\text{BS}}_{KT}}{dS}$$

- Once hedge portfolio broken down into underlying + *naked* vanilla options, delta always equal to $\Delta^{\text{MM}} = \left.\frac{d\mathcal{P}}{dS}\right|_{O_{KT}}$.
- Nothing fundamental about Δ^{SS} – tied to a particular representation of vanilla option prices.

Dynamics in LV model

- What's left before we can use LV model? Output the ν_{KT}, see if we like them.
  - More practical to look at implied vols for floating strike – fixed moneyness.
- Look at vols of vols and spot/vol covariances.
- For ATMF vol equivalently look at SSR $\mathcal{R}_T$

$$\mathcal{R}_T = \frac{1}{\mathcal{S}_T}\frac{\langle d\widehat{\sigma}_{F_T T}\; d\ln S\rangle}{\langle (d\ln S)^2\rangle} = \frac{1}{\mathcal{S}_T}\frac{d\widehat{\sigma}_{F_T T}}{d\ln S}$$

$$\mathcal{S}_T = \left.\frac{d\widehat{\sigma}_{KT}}{d\ln K}\right|_{F_T}$$

$$\text{vol}(\widehat{\sigma}_{F_T T}) = \mathcal{R}_T\left(\mathcal{S}_T\frac{\widehat{\sigma}_{F_0 0}}{\widehat{\sigma}_{F_T T}}\right)$$

- Assume following expression for LV function:

$$\sigma(t,S) = \sigma(t) + \alpha(t)x + \frac{\beta(t)}{2}x^2, \quad x = \ln\left(\frac{S}{F_t}\right)$$

and calculate $\mathcal{S}_T$, $\mathcal{R}_T$ at order 1 in $\alpha(t), \beta(t)$.

Dynamics in LV model – 2

- Use variance $u = \sigma^2$ and write $u(t,S) \;=\; u_0(t) + \delta u(t,S)$. At order 1 in δu:

$$\widehat{\sigma}^2_{KT} \;=\; \frac{1}{T}\int_0^T dt \int_{-\infty}^{+\infty} dy\, \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, u\left(t, F_t e^{\frac{\omega_t}{\omega_T}x_K + \frac{\sqrt{(\omega_T-\omega_t)\omega_t}}{\sqrt{\omega_T}}y}\right)$$

where $x_K = \ln(\frac{K}{F_T})$ and $\omega_t \;=\; \int_0^t u_0(\tau)d\tau$.

- Expanding around a cst $\sigma(t) = \sigma_0$: $u_0 = \sigma_0^2$

$$\widehat{\sigma}_{KT} \;=\; \frac{1}{T}\int_0^T dt \int_{-\infty}^{+\infty} dy\, \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, \sigma\left(t, F_t e^{\frac{t}{T}x_K + \sigma_0\sqrt{\frac{(T-t)t}{T}}y}\right)$$

- From this get:

$$\mathcal{S}_T \;=\; \left.\frac{d\widehat{\sigma}_{KT}}{d\ln K}\right|_{K=F_T} \;=\; \frac{1}{T}\int_0^T \frac{t}{T}\alpha(t)dt \qquad \text{"skew averaging" – V. Piterbarg}$$

$$\left.\frac{d^2\widehat{\sigma}_{KT}}{d\ln K^2}\right|_{K=F_T} \;=\; \frac{1}{T}\int_0^T \left(\frac{t}{T}\right)^2 \beta(t)dt$$

$$\left.\frac{d\widehat{\sigma}_{KT}}{d\ln S}\right|_{K=F_T} \;=\; \frac{1}{T}\int_0^T \left(1-\frac{t}{T}\right)\alpha(t)dt$$

Dynamics in LV model – 3

- From 1st equation: $\alpha\left(t\right) \;=\; \frac{d}{dt}\left(t\mathcal{S}_t\right) + \mathcal{S}_t.$

$$\frac{d\widehat{\sigma}_{F_T(S)T}}{d\ln S} \;=\; \left(\left.\frac{d\widehat{\sigma}_{KT}}{d\ln K}\right|_{K=F_T} + \left.\frac{d\widehat{\sigma}_{KT}}{d\ln S}\right|_{K=F_T} \right) \;=\; \frac{1}{T}\int_0^T \alpha(t)dt \;=\; \mathcal{S}_T + \frac{1}{T}\int_0^T \mathcal{S}_t dt$$

- Get expression of SSR: $\mathcal{R}_T \;=\; \frac{1}{\mathcal{S}_T}\frac{\left\langle d\widehat{\sigma}_{F_T T}\;\; d\ln S\right\rangle}{\left\langle (d\ln S)^2\right\rangle} \;=\; \frac{1}{\mathcal{S}_T}\frac{d\widehat{\sigma}_{F_T(S)T}}{d\ln S}$:

$$\mathcal{R}_T \;=\; 1 + \frac{1}{T}\int_0^T \frac{\mathcal{S}_t}{\mathcal{S}_T}dt$$

- Limiting behavior
  - short maturities:

$$\lim_{T\to 0}\mathcal{R}_T \;=\; 2 \quad \text{OK}$$

  - long maturities – take $\mathcal{S}_T \propto \frac{1}{T^\gamma}$:

$$\lim_{T\to\infty}\mathcal{R}_T \;=\; \frac{2-\gamma}{1-\gamma}$$

- For typical value $\gamma = \frac{1}{2}$, $\lim_{T\to\infty}\mathcal{R}_T \;=\; 3.$

Dynamics in LV model – 4

- What about expanding around a deterministic volatility $\sigma_0(t)$ rather than a cst σ_0?

$$\mathcal{S}_T = \left.\frac{d\widehat{\sigma}_{KT}}{d\ln K}\right|_{K=F_T} = \frac{1}{T}\int_0^T \frac{\widehat{\sigma}_t^2 t}{\widehat{\sigma}_T^2 T}\frac{\sigma_0(t)}{\widehat{\sigma}_T}\alpha(t)dt$$

$$\left.\frac{d\widehat{\sigma}_{KT}}{d\ln S}\right|_{K=F_T} = \frac{1}{T}\int_0^T \left(1-\frac{\widehat{\sigma}_t^2 t}{\widehat{\sigma}_T^2 T}\right)\frac{\sigma_0(t)}{\widehat{\sigma}_T}\alpha(t)dt$$

$$\frac{d\widehat{\sigma}_{F_T T}}{d\ln S} = \mathcal{S}_T + \frac{1}{T}\int_0^T \frac{\sigma_0^2(t)}{\widehat{\sigma}_t\widehat{\sigma}_T}\mathcal{S}_t dt$$

where $\widehat{\sigma}_t = \sqrt{\frac{1}{t}\int_0^t \sigma_0^2\left(u\right)du}$.

- SSR given by:

$$\mathcal{R}_T = 1 + \frac{1}{T}\int_0^T \left(\frac{\sigma_0^2(t)}{\widehat{\sigma}_t\widehat{\sigma}_T}\right)\frac{\mathcal{S}_t}{\mathcal{S}_T}dt$$

- Better approx of $\mathcal{R}_T$ for strongly sloping term structures of ATMF volatilities.

Dynamics in LV model – 5

- Check approx of SSR on 2 smiles of Eurostoxx50

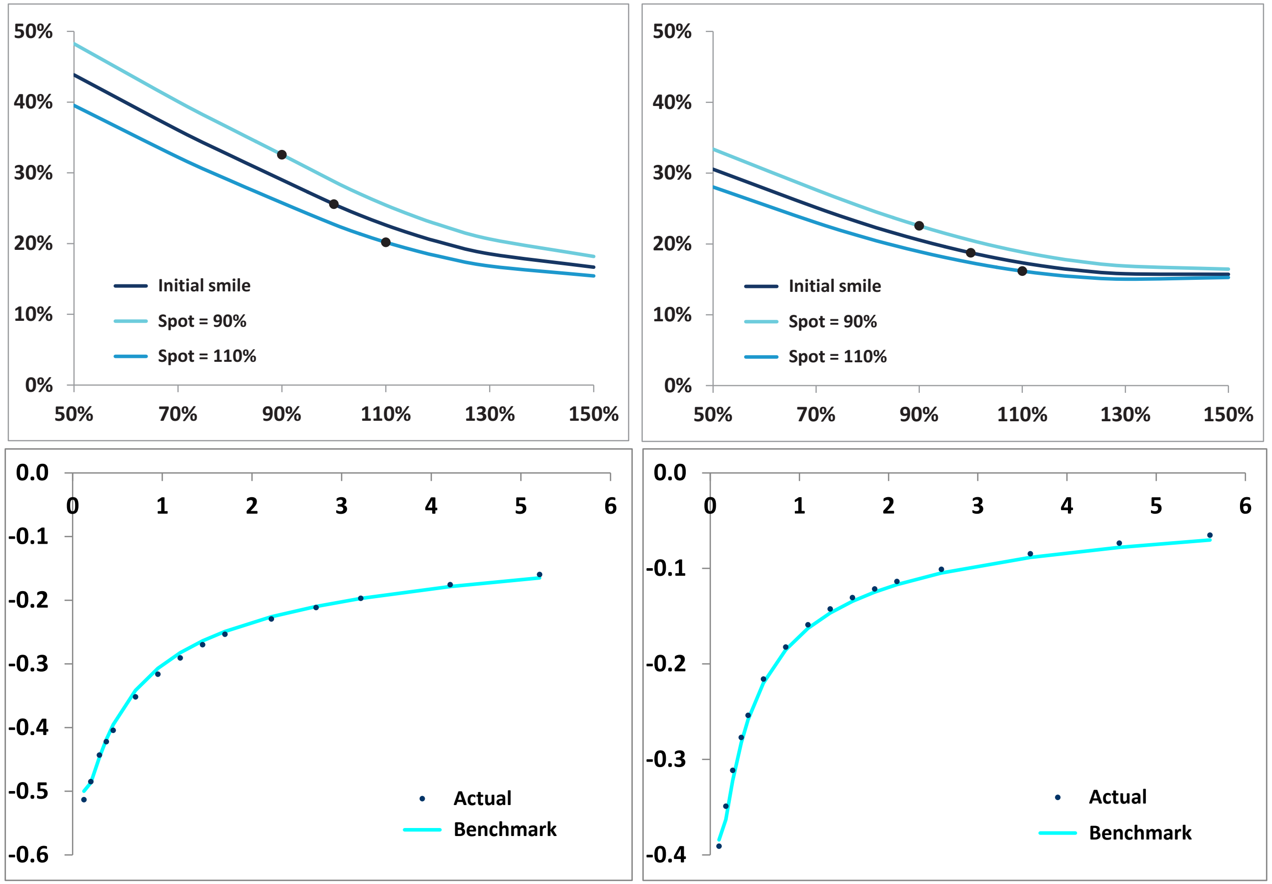


Figure: Top: smiles of the Eurostoxx50 index for a maturity $\simeq 1$ year observed on October 4, 2010 (left) and May 16, 2013 (right). Bottom: term structures of ATMF skew and power-law fits with $\gamma = 0.37$ (left), $\gamma = 0.52$ (right), as a function of T (years).

Dynamics in LV model – 6

- Real versus approximate SSR

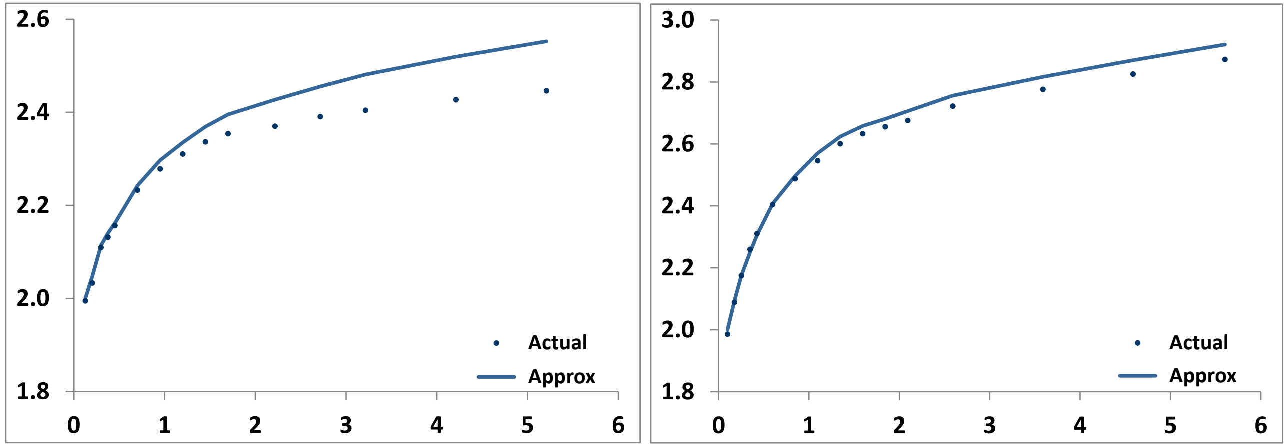
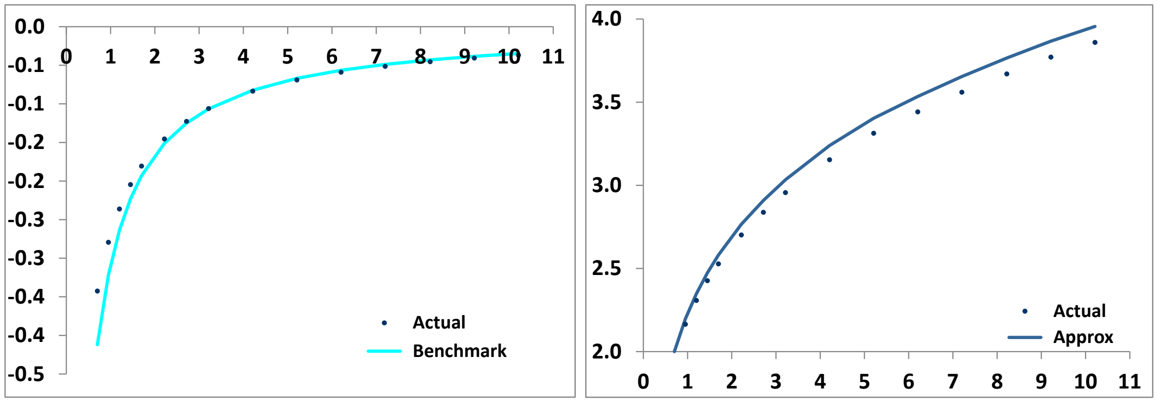


Figure: $\mathcal{R}_T$ as a function of T (years) computed: (a) in FD (actual), (b) using expression $\mathcal{R}_T \;=\; 1 + \frac{1}{T}\int_0^T \frac{\mathcal{S}_t}{\mathcal{S}_T}\,dt$ (approx).

- What about smile with $\mathcal{S}_T \propto \frac{1}{T}$? Approx fomula gives $\lim_{T\to\infty}\mathcal{R}_T = \infty$ (logarithmic divergence of $\mathcal{R}_T$):



- Approx slightly overestimates SSR.

Conclusion

- LV model is a genuine market model for underlying + vanilla options
  ... that happens to possess a 1-d Markov representation in terms of (t,S).

- Generates well-defined break-even levels for spot/vol and vol/vol covariances in the carry P&L.

- Daily recalibration of LV function – an ancillary object – is exactly how model should be used and deltas calculated.
  - Spot/vol break-even correlations $= -100\%$, vol/vol break-even correlations $= 100\%$.
  - Volatilities of implied volatilities given by: $\text{vol}(\widehat{\sigma}_{KT}) \;=\; \frac{1}{\Sigma^{\text{LV}}_{KT}} \frac{d\Sigma^{\text{LV}}_{KT}}{dS} S\sigma(t,S)$.

- Delta is well-defined: $\Delta^{\text{MM}} \;=\; \left.\frac{d\mathcal{P}}{dS}\right|_{O_{KT}}$. Delta of vanilla option irrelevant notion.

- When vega-hedging with (BS) delta-hedged vanilla options, sticky-strike delta should be used.

- Good approximate formula for sizing up break-even vols of ATMF vols – or equivalently SSR:

$$
\begin{aligned}
\mathcal{R}_T &= 1 + \frac{1}{T}\int_0^T \frac{\mathcal{S}_t}{\mathcal{S}_T}\,dt \\
\text{vol}(\widehat{\sigma}_{F_T T}) &= \mathcal{R}_T \left(\mathcal{S}_T \frac{\widehat{\sigma}_{F_0 0}}{\widehat{\sigma}_{F_T T}}\right)
\end{aligned}
$$