What generates equity smiles ?

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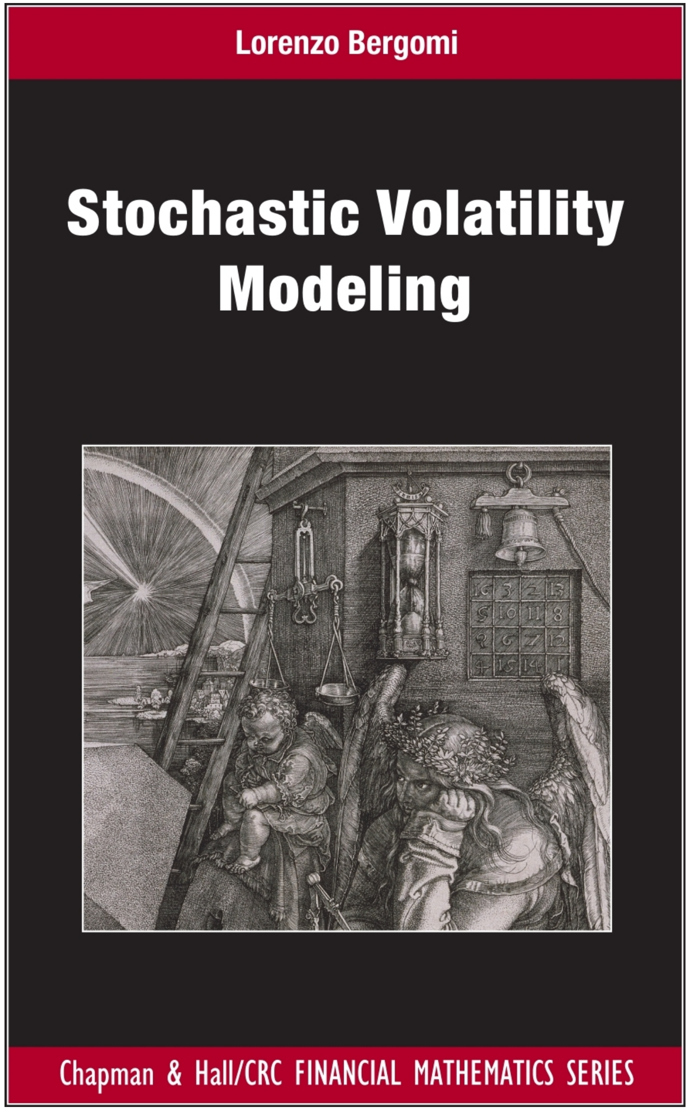
Outline

- Sources of the equity smile ?
- Historical distribution of daily returns
- An SV model with conditional non-Gaussian returns
- Impact on vanilla smiles
- Impact on path-dep payoffs



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Intro – 1

- What generates equity smiles? Supply & demand
- OK, but what should the "fair" smile look like?

- If no vanillas exist - have to quote one
  - Delta-hedge vanilla ⇨ gamma/theta P&L
  - "Fair" ATMF skew given by covariance of spot & future realized variance

- If ATM vanillas liquid
  - Consider call spread position centered on ATMF such that $\Gamma = 0$
  - Initially $\Gamma = 0$, but $\Gamma+/\Gamma-$ when spot moves
  - Trade dynamically ATMF vanillas to cancel Γ
  - Carry P&L: spot/ATMF vol cross-gamma + ATMF vol gamma. Latter risk smaller

⇨ In liquid markets ATMF skew measures implied level of spot/ATMF vol covariance (not spot/future realized vol)

- At order 1 in vol-of-vol:

$$\mathcal{S}_T = \left.\frac{d\widehat{\sigma}_{KT}}{d\ln K}\right|_F = \frac{1}{2\widehat{\sigma}_T^3 T}\int_0^T \frac{T-t}{T}\left\langle d\ln S_t \; d\widehat{\sigma}_{T,t}^2\right\rangle$$

$\widehat{\sigma}_{T,t}$: ATMF/VS vol at t for maturity T

Intro – 2

- So far considered P&L at order 2 in δS, $\delta\widehat{\sigma}_T$
- What about large spot moves ?
  - Responsible for steep smiles ?
- Do large drawdowns generate a significant portion of the vanilla smile ?
- Do they impact other derivatives ?

Unconditional distribution of daily returns – 1

- Take 1 century worth of DJIA daily closes[1] ⇨ daily returns $r_i = \frac{S_i}{S_{i-1}} - 1$
- Separately normalize positive/negative returns:

$$r_i \quad \rightarrow \quad \overline{r}_i = \frac{r_i}{\sqrt{\langle r_i^2 \rangle}}$$

- Rank negative returns from lowest to highest. Define empirical cumulative density of normalized negative returns as:

$$P\left[\overline{r} \leq \overline{r}_i\right] \;=\; \frac{1}{2}\,\frac{i}{N^-}$$

- Graph $log_{10} P[\overline{r} \leq r]$. Do same for positive returns
- Compare with (a) lognormal distribution, (b) Student distribution:

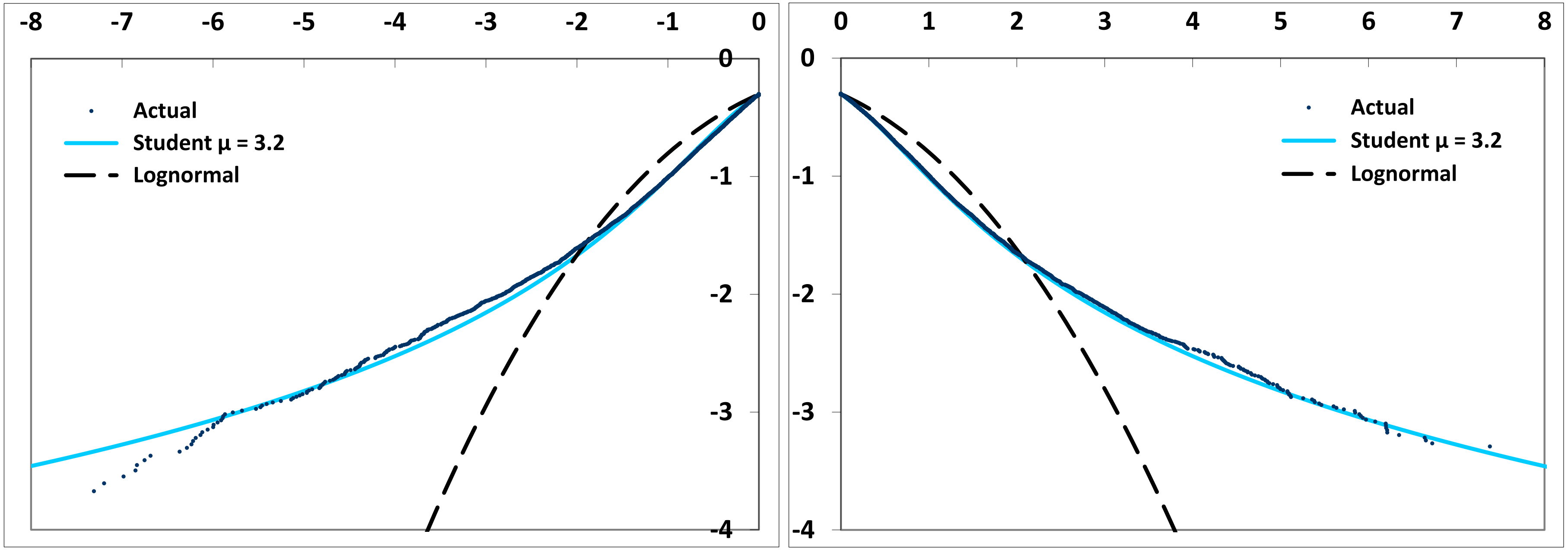
$$\rho_\mu\left(\overline{r}\right) \;=\; \frac{\Gamma\left(\frac{1+\mu}{2}\right)}{\sqrt{\mu\pi}\;\Gamma\left(\frac{\mu}{2}\right)} \frac{1}{\left(1+\frac{\overline{r}^2}{\mu}\right)^{\frac{1+\mu}{2}}} \qquad \propto \; \frac{1}{|\overline{r}|^{1+\mu}} \;\; \text{for } \overline{r} \text{ large}$$

μ: number of degrees of freedom

[1] Available on *http://stooq.com*. Present sample: [Jan 1st 1900, July 20, 2014]

Unconditional distribution of daily returns – 2

- Dow Jones: 1900-2014



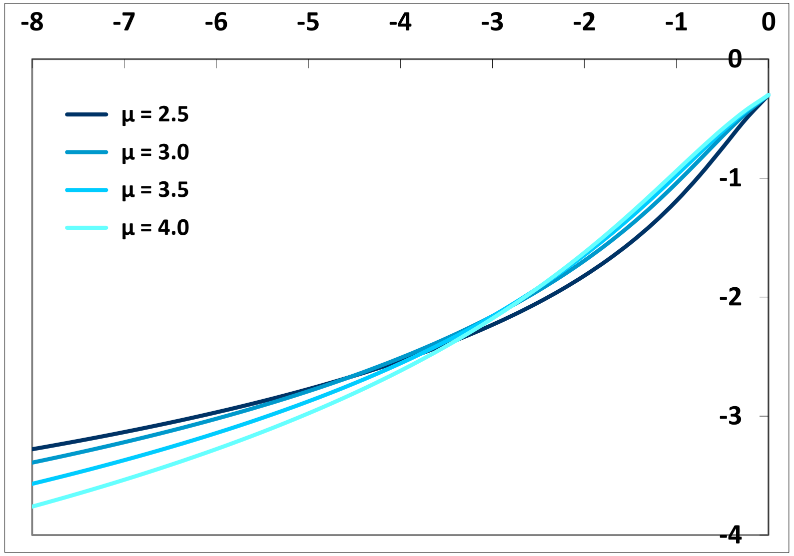
  - 14103 negative returns. $\min(\overline{r}_i) = -19.6$. About 100 values of $\overline{r}_i \leq -4$.
  - 15657 positive returns. $\max(\overline{r}_i) = 14.3$.
  - Fit with $\mu = 3.2$

- Student distribution
  - The smaller μ the thicker the tails. Only moments of order $< \mu$ exist
  - Variance $= \frac{\mu}{\mu - 2}$. Kurtosis $= \frac{6}{\mu - 4}$.
  - For $\mu \to \infty$ converges to Gaussian distribution

⇨ Fit OK

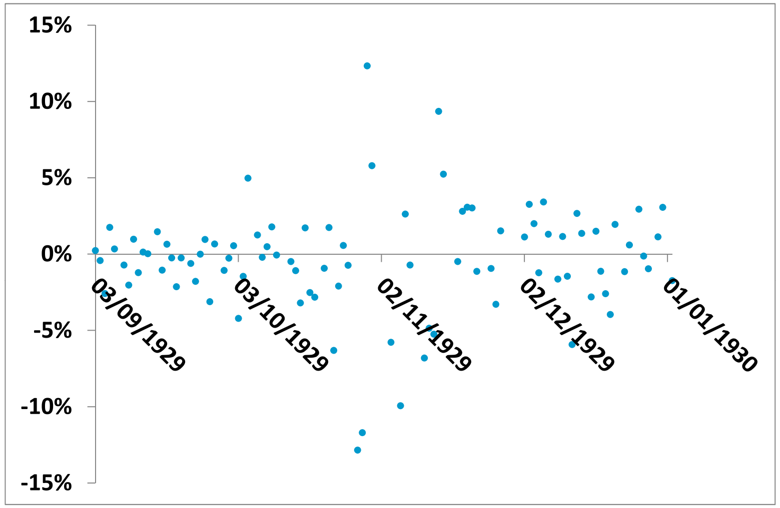
Unconditional distribution of daily returns – 3

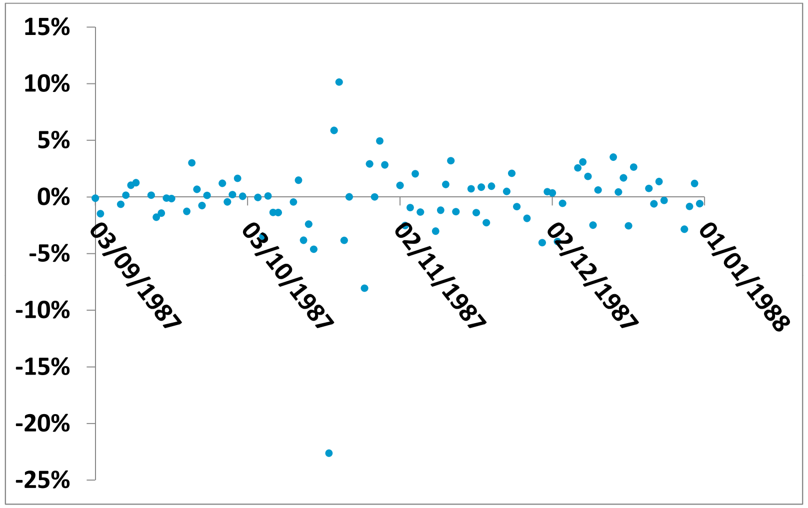
- Cumulative densities for different values of μ, all with $E[\bar{r}^2] = 1$.



⇨ $\mu \in [3, 4]$ acceptable

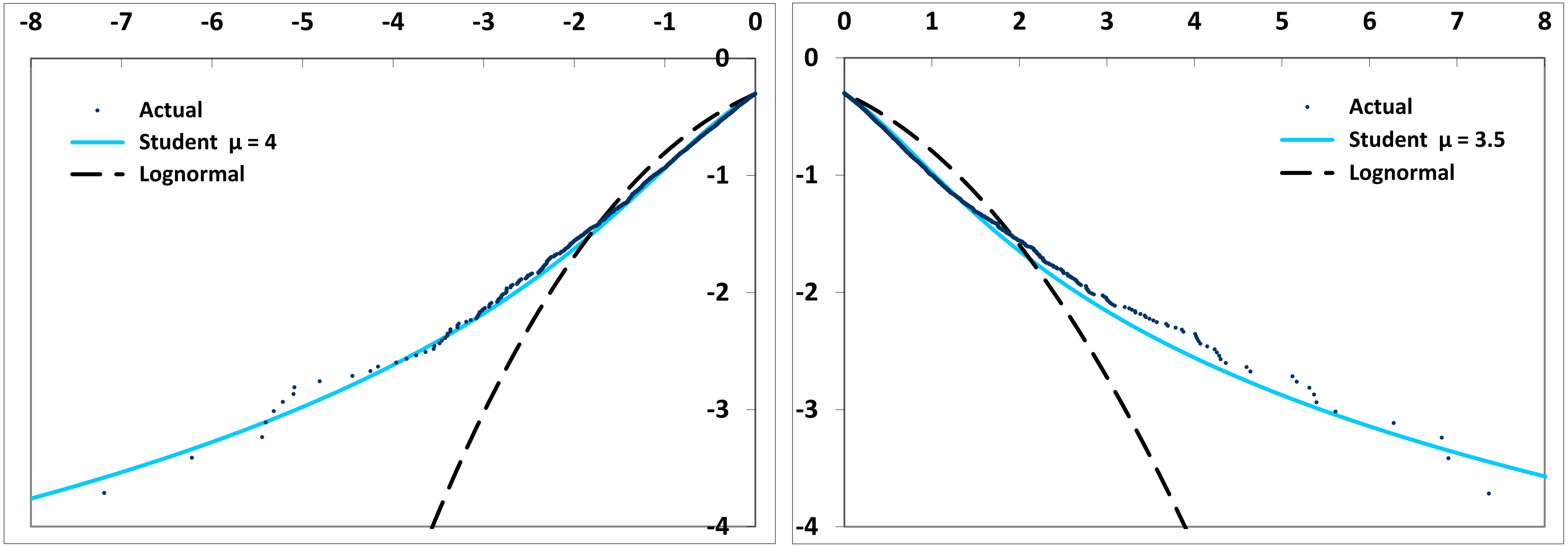
- Left/right tails of empirical density similar ?? Yes
  - Dow Jones daily returns: Sep–Dec 1929 / Sep–Dec 1987





Unconditional distribution of daily returns – 4

- Other example: HSCEI index, 1993-2014



⇨ Daily returns of equity indexes well captured by Student distribution

⇨ So far have looked at unconditional distribution – lumps together very different volatility regimes.

- Write

$$r_i \;=\; \sigma_i \sqrt{\delta t}\; z_i \qquad E[z_i^2] = 1$$

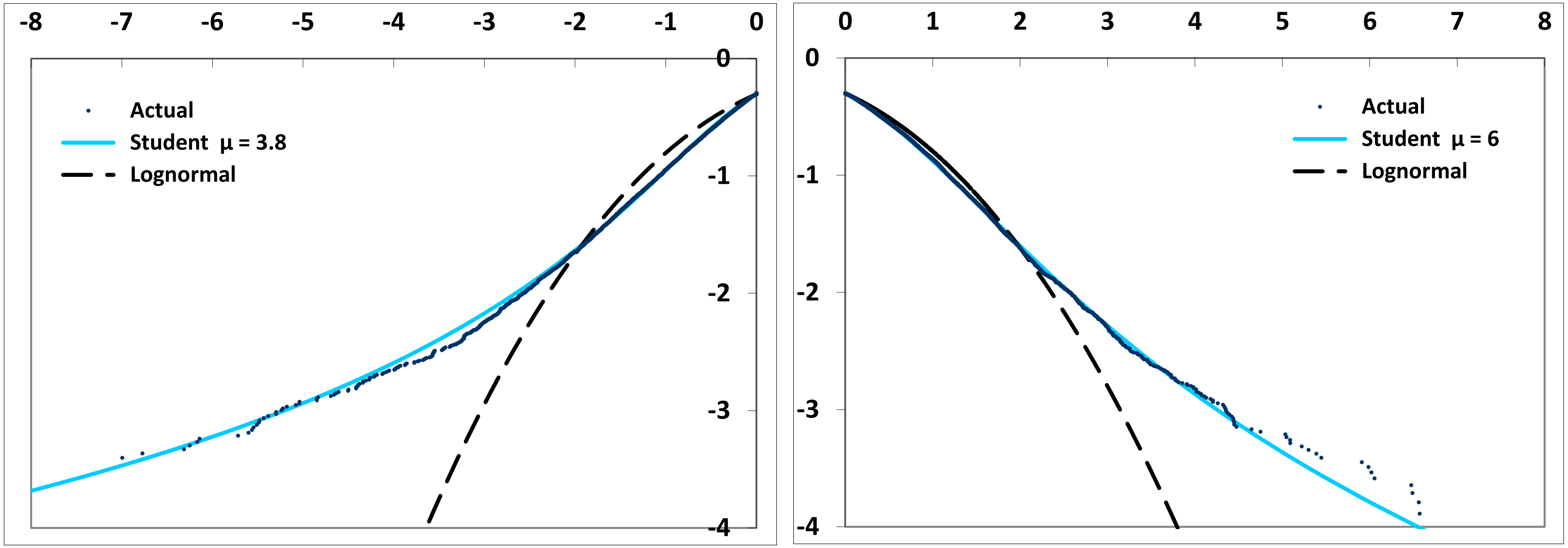
- Fat tails of r_i due to randomness of σ_i ?
- Look at conditional distribution

Conditional distribution of daily returns

$$r_i = \sigma_i \sqrt{\delta t}\; z_i$$

- No easy acces to σ_i – unless intraday data available.
  - Intrinsic noise of estimator of σ_i pollutes estimation of tails of z_i
- Proxy for σ_i: 1-year realized vol

- Dow Jones: 1900-2014 again



  - μ larger than in unconditional distribution: OK

⇨ Even accounting for randomness of volatility, daily returns are fat-tailed

⇨ z_i markedly non-Gaussian ⇨ sizeable 1-day conditional smile

⇨ How does the 1-day conditional smile impact derivatives?

SV model with conditional 1-day smile

- 1-day smile generates higher-order contributions to carry P&L, beyond gammas & cross-gammas of spot/implied vols

⇨ Need SV model to assess impact of (unhedgeable) 1-day smile risk

  - SV part of model sets σ_i, i.e. scale of daily returns
  - 1-day smile params govern 1-day conditional density
  - ... while keeping (a) vols of (implied) vols, (b) covariances of spot & (implied) vols unchanged.

- Start with 2-factor fwd variance workhorse

  - Dynamics of inst. fwd variances ξ_t^T:[2]

$$\frac{d\xi_t^T}{\xi_t^T} = (2\nu)\,\alpha\left((1-\theta)e^{-k_1(T-t)}dW_t^X + \theta e^{-k_2(T-t)}dW_t^Y\right)$$

  - Curve ξ_t^T a function of two OU processes X_t, Y_t:

$$\xi_t^T = \xi_0^T e^{(2\nu)\alpha\left[(1-\theta)e^{-k_1(T-t)}X_t \,+\, \theta e^{-k_2(T-t)}Y_t\right] - \frac{4\nu^2\alpha^2}{2}\bullet}$$

$$dX_t = -k_1 X_t dt + dW_t^X \qquad dY_t = -k_2 Y_t dt + dW_t^Y$$

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[2] α normalization factor: $\alpha = 1/\sqrt{(1-\theta)^2 + \theta^2 + 2\rho_{XY}\theta\,(1-\theta)}$ ⇨ $\text{vol}(\xi_t^t) = 2\nu$ ⇨ $\text{vol}(\sqrt{\xi_t^t}) = \nu$

SV model – 1

- Process for S_t:

$$dS_t = (r-q)S_t dt + \sqrt{\xi_t^t} S_t dW_t^S$$

$$\langle dW^S dW^X \rangle = \rho_{SX} dt \qquad \langle dW^S dW^Y \rangle = \rho_{SY} dt$$

⇨ Instantaneous vol of VS vol $\widehat{\sigma}_T$ of maturity T – for flat TS of VS vols:

$$\text{vol}(\widehat{\sigma}_T) = \nu\, \alpha \sqrt{(1-\theta)^2\, I^2(k_1 T) + \theta^2 I^2(k_2 T) + 2\rho_{XY}\theta(1-\theta)\, I(k_1 T)\, I(k_2 T)}$$

$$I(x) = \frac{1-e^{-x}}{x}$$

$$\text{vol}(\widehat{\sigma}_{T=0}) = \nu$$

- Vol of ATMF vol ≈ vol of VS vol

⇨ ATMF skew at order 1 in vol-of-vol for flat TS of VS vols:

$$\mathcal{S}_T = \nu\alpha \left[(1-\theta)\, \rho_{SX} \frac{k_1 T - \left(1 - e^{-k_1 T}\right)}{(k_1 T)^2} + \theta \rho_{SY} \frac{k_2 T - \left(1 - e^{-k_2 T}\right)}{(k_2 T)^2} \right]$$

SV model – 2

- Params $\theta, k_1, k_2, \rho_{XY}$? ⇨ so that vol($\widehat{\sigma}_T$) matches power-law benchmark

$$\mathrm{vol}(\widehat{\sigma}_T)_{\mathrm{Benchmark}} = \nu_0 \left(\frac{3\mathrm{m}}{T}\right)^{\alpha}$$

over range $[T_{\min}, T_{\max}]$.

- Typically, $\alpha = 0.4$, $\nu_0 = 60\%$ (realized) / 100% (implied)
  - Sets for $\alpha = 0.4$, $\nu_0 = 100\%$, range $[1\mathrm{m}, 5\mathrm{y}]$

ν	120.9%	135.8%	174.0%	178.2%	181.9%	185.1%	190.1%
θ	57.9%	30.1%	24.5%	23.8%	23.4%	23.1%	22.8%
k_1	0.58	2.59	5.35	6.02	6.65	7.26	8.34
k_2	1.19	0.32	0.28	0.27	0.25	0.24	0.22
ρ_{XY}	-95%	-50%	0%	20%	40%	60%	99%

  - No over-parametrization. Different sets ⇨ different short vol/long vol correlations

- Spot/factor correls ρ_{SX}, ρ_{SY} such that:
  - Either generate given level & term-structure of covariances of spot & ATMF vols
  - Or generate desired term structure of ATMF skew. Typically:

$$\mathcal{S}_T \propto \frac{1}{T^{\gamma}} \quad \text{with} \quad \gamma \in [0.3, 0.7], \ \text{range } [1\mathrm{m}, 5\mathrm{y}]$$

SV model with conditional 1-day smile – 1

- Set time scale $\Delta = 1$ day
- Fwd variances: simulate increments $\delta X, \delta Y$ of OU processes X, Y

$$\delta X = \int_t^{t+\Delta} e^{-k_1(t+\Delta-u)} dW_u^X \qquad \delta Y = \int_t^{t+\Delta} e^{-k_2(t+\Delta-u)} dW_u^Y$$

- Spot increment:

$$S_{t+\Delta} = S_t \left[1 + (r-q)\,\Delta + \sigma_t\,\delta Z \right]$$

with

$$\sigma_t = \sqrt{\frac{1}{\Delta} \int_t^{t+\Delta} \xi_t^\tau \, d\tau} \approx \sqrt{\xi_t^t} \text{ if } \Delta \text{ small}$$

⇨ In standard 2F model

$$S_{t+\Delta} = S_t \left[1 + (r-q)\,\Delta + \sigma_t\,\delta W^S \right]$$

⇨ Here δZ fat-tailed, no longer Gaussian

SV model with conditional 1-day smile – 2

- δZ: 2-sided Student distribution with params μ_+, μ_-

- Histo. positive & negative returns $\approx$ equally probable ⇨ 1-day ATM digital $\approx \frac{1}{2}$
- In model, want ability to set 1-day ATM skew at will

⇨ Introduce p^+, p^-: probabilities of positive/negative returns

$$\begin{cases} \delta Z = \sigma_+ \sqrt{\Delta}\, |X_{\mu_+}| & \text{with probability } p_+ \\ \delta Z = -\sigma_- \sqrt{\Delta}\, |X_{\mu_-}| & \text{with probability } p_- \end{cases}$$

X_μ: Student random variable with μ degrees of freedom

⇨ σ_+, σ_- such that $E[\delta Z] = 0$, $E[\delta Z^2] = \Delta$

- Need to correlate δZ with $\delta X, \delta Y$

SV model with conditional 1-day smile – 3

⇨ Define function f that maps Brownian increment δW^S into δZ:

$$\frac{\delta Z}{\sqrt{\Delta}} = f\left(\frac{\delta W^S}{\sqrt{\Delta}}\right)$$

$$\begin{cases} x \le \mathcal{N}_G^{-1}(p_-): & f(x) = \zeta_- \sqrt{\frac{\mu_- - 2}{\mu_-}}\, \mathcal{N}_{\mu_-}^{-1}\left(\frac{\mathcal{N}_G(x)}{2p_-}\right) \\ x \ge \mathcal{N}_G^{-1}(p_-): & f(x) = \zeta_+ \sqrt{\frac{\mu_+ - 2}{\mu_+}}\, \mathcal{N}_{\mu_+}^{-1}\left(\frac{1}{2} + \frac{\mathcal{N}_G(x) - p_-}{2p_+}\right) \end{cases}$$

- $\mathcal{N}_G$: CDF of standard normal variable, $\mathcal{N}_G^{-1}$ its inverse
- $\mathcal{N}_\mu^{-1}$: inverse CDF of Student random variable with μ degrees of freedom
- ζ^+, ζ^- given by:

$$\zeta_+ = \frac{p_-\alpha_-}{\sqrt{p_+(p_-\alpha_-)^2 + p_-(p_+\alpha_+)^2}} \qquad \zeta_- = \frac{p_+\alpha_+}{\sqrt{p_+(p_-\alpha_-)^2 + p_-(p_+\alpha_+)^2}}$$

with $\alpha_+ = \frac{2}{\sqrt{\pi}} \frac{\sqrt{\mu_+ - 2}}{\mu_+ - 1} \frac{\Gamma\left(\frac{1+\mu_+}{2}\right)}{\Gamma\left(\frac{\mu_+}{2}\right)}$ and likewise for α_-

⇨ Mapping function f built once and for all.

$$E[f(x)] = 0 \quad E[f^2(x)] = 1$$

SV model with conditional 1-day smile – 4

- Now able to generate δZ from Brownian increment δW^S: $\frac{\delta Z}{\sqrt{\Delta}} = f\left(\frac{\delta W^S}{\sqrt{\Delta}}\right)$
- Last part of job: correlate δW^S with δX, δY
  - Covariances $E[\delta Z \delta X]$, $E[\delta Z \delta Y]$ need to stay fixed
- In fat-tailed version of model, use correlations ρ^*_{SX}, ρ^*_{SY} such that:

$$E_*\left[\delta Z \delta X\right] = E\left[\delta W^S \delta X\right] \quad \text{likewise for } E\left[\delta Z \delta Y\right]$$

  - Standard 2F model: $\delta X = I(k_1 \Delta)\left(\rho_{SX} \delta W^S + \dots\right)$ $\qquad I(x) = \frac{1-e^{-x}}{x}$
  - Fat-tailed 2F model: $\delta X = I(k_1 \Delta)\left(\rho^*_{SX} \delta W^S + \dots\right)$
  - Equate covariance of δX with $\delta Z, \delta W^S$:

$$E_*\left[\delta Z\left(\bullet\, \rho^*_{SX} \delta W^S\right)\right] = E\left[\delta W^S\left(\bullet\, \rho_{SX} \delta W^S\right)\right] = \bullet\, \rho_{SX} \Delta$$

- Yields:

$$\frac{\rho^*_{SX}}{\rho_{SX}} = \frac{\Delta}{E\left[\delta Z \delta W^S\right]} = \frac{1}{\int \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, x\, f(x) dx}$$

⇨ Rescaling of spot/vol correlations same for all factors:

$$\frac{\rho^*_{SY}}{\rho_{SY}} = \frac{\rho^*_{SX}}{\rho_{SX}} \geq 1 \quad \text{(Cauchy-Schwarz)}$$

SV model with conditional 1-day smile – 5

- Fait-tailed 2F model
  - Standard simulation of 2 OU processes X, Y with correls ρ^*_{SX}, ρ^*_{SY} with W^S.
  - Spot simulation: no harder than in standard 2F model:

$$S_{t+\Delta} \;=\; S_t\left[(r-q)S_t\Delta + \sigma_t\sqrt{\Delta}f\left(\frac{\delta W^S}{\sqrt{\Delta}}\right)\right]$$

⇨ Pricing time similar to 2F standard model – in practice $\sigma_t = \sqrt{\xi^t_t}$

⇨ Can vary 1-day mile (i.e. f) while leaving dynamics of vols unchanged: vols of implied vols, correls of spot & implied vols

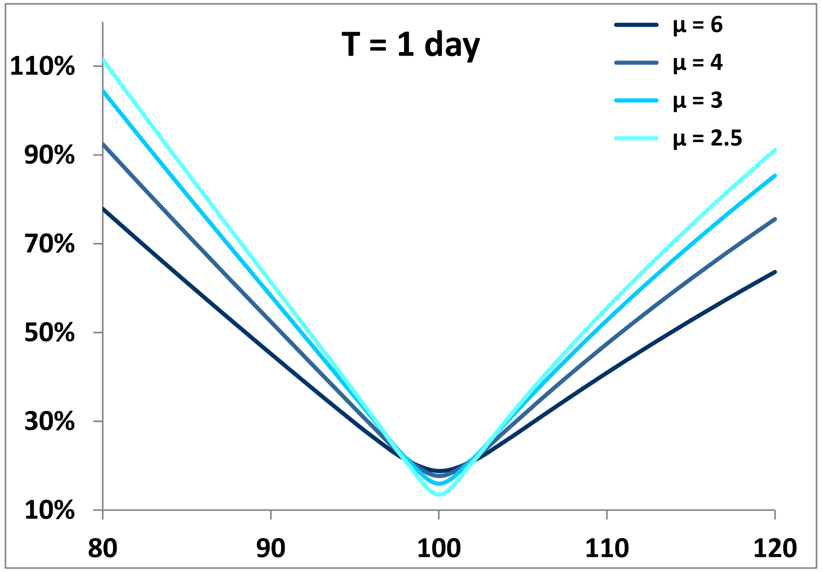
- 1-day smile params only change conditional density of normalized daily returns
- Neither possible with jump/diffusion, nor with time-changed Lévy processes L_{τ_t}
  - Conditional skewness & kurtosis fixed, correlation of spot and vols fixed
- Continuous limit of model ?? Depends on scaling of $p^+(\Delta) - \frac{1}{2}$ and $\mu(\Delta)$ as $\Delta \to 0$.

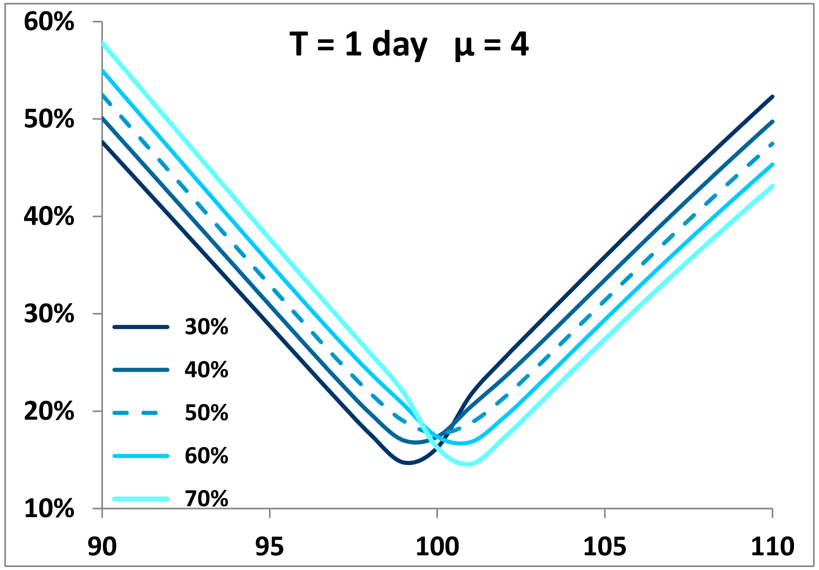
1-day smile

- Left: $p_+ = p_- = \frac{1}{2}$. Right: $p_+ \neq p_-$.

  $\mu_+ = \mu_-$, $\sigma_t = 20\%$.

- Smile is obtained by numerical integration





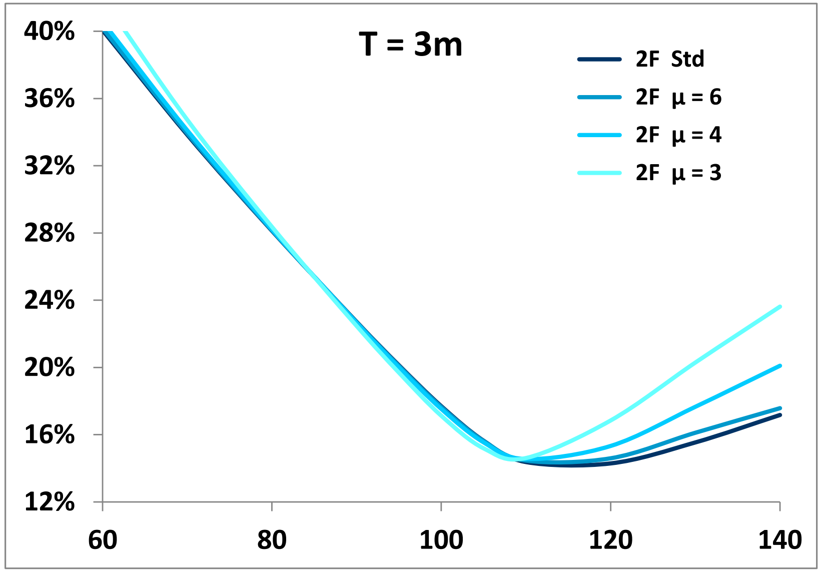
⇨ p_+, μ_+, μ_- do what they're supposed to do.

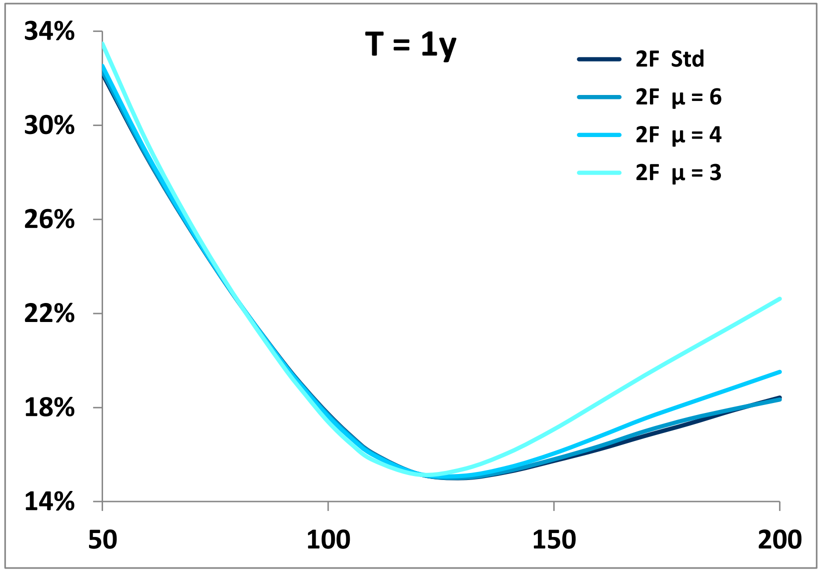
Vanilla smile – 1

- Parameters of 2F model (typical of STOXX50 – July 2014)

ν	θ	k_1	k_2	ρ_{XY}	ρ_{SX}	ρ_{SY}
257%	15.1%	8.96	0.46	40%	-74.6%	-13.7%

- 3m and 1y smiles for different $\mu_+ = \mu_-$, $p_+ = 0.5$, VS vols flat at 20%.





Std: standard 2F model – equivalent to $\mu = \infty$

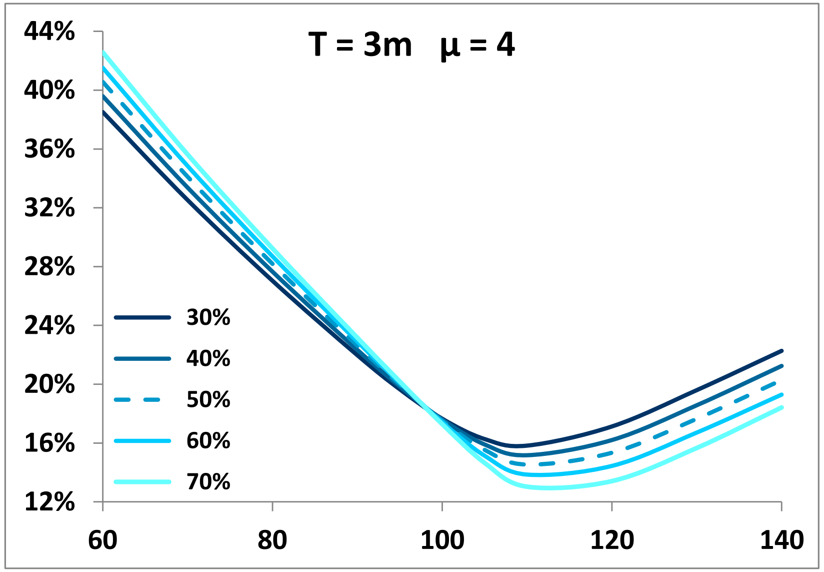
- Std 2F model diffusive: algos for quasi-real-time vanilla smile generation – see book
- Fat-tailed 2F model is not: really have to price (delta-hedged) call/put payoffs

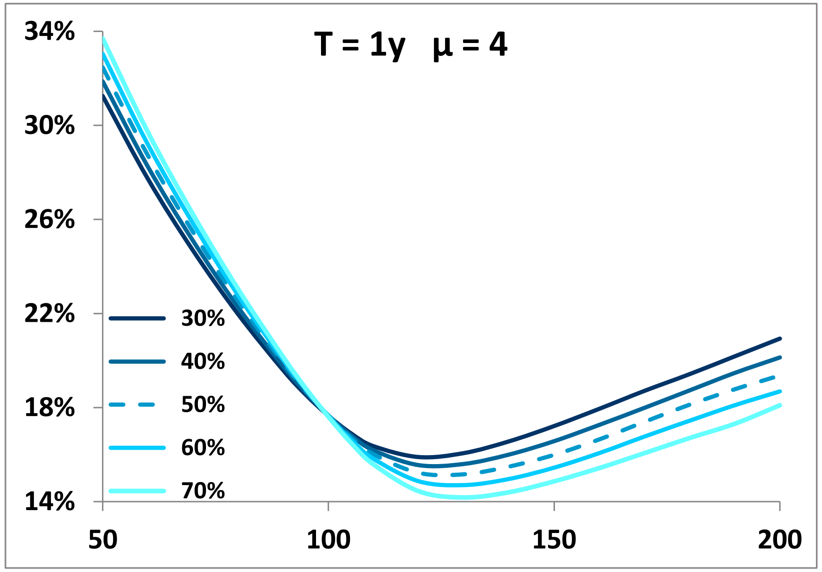
⇨ 1-day smile has minute impact on vanilla near-ATM smile

⇨ 1-day smile impacts tails – mostly OTM calls (for equities)

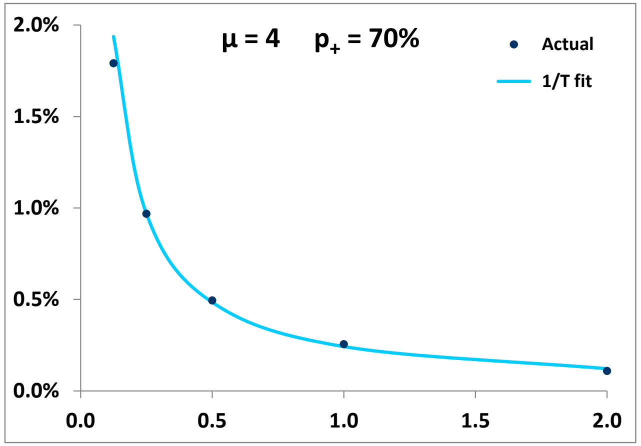
Vanilla smile – 2

- Impact of 1-day ATM digit. $\mu_+ = \mu_- = 4$.





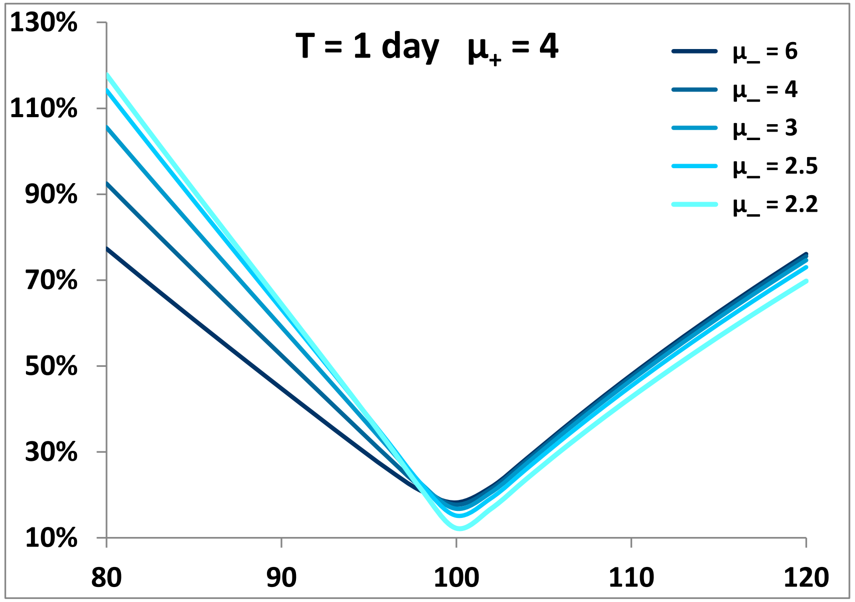
- Scaling of 1-day skew contribution to 95/105 skew. Turn off stoch vol: $\nu = 0$



⇨ Contribution of 1-day skew to vanilla ATM skew $\propto \frac{1}{T}$: OK

Example 1: Daily cliquets – Gap notes – Crash puts

- Similar to CDS contract. Maturities 3m, 6m, 1y
- Receive Put (90%, 80%, 75%) or Put spread (90%/80%, 85%/75%) payoff on daily index returns
- Pay quarterly spread, starting at inception. Expires when 1st Put/Put spread is triggered
  - No delta, some vega – almost pure 1-day smile payoff
- Left: 1-day smile for different values of μ_-. $\mu_+ = 4, p_+ = 0.5$, vol $= 20\%$
- Right: upfront prices for 1-year 80% Crash Put – in basis points



μ_-	∞	6	4	3	2.5	2.2
v = 0%	0	0	2	15	43	62
v = 257%	1	3	9	28	53	67

⇨ Market prices very conservative, correspond to implied value of $\mu_- \approx 2.2$

Example 2: Var swaps

- $\ln(\frac{S_{i+1}}{S_i})^2$: VS other instance of daily cliquet
  - Assume no dividends. Consider position: short VS/long vanilla replication of $-2\ln S$, delta-hedged
  - Carry P&L cancels up to order 2 in δS.
  - Contribution of higher orders ⇨ $\widehat{\sigma}_{VS} \neq \widehat{\sigma}_{Logswap}$
- $(\widehat{\sigma}_{VS} - \widehat{\sigma}_{Logswap})$ for 1-year maturity, with/without stoch vol, for $p_+ = \frac{1}{2}$.

μ	∞	6	4	3
ν = 0	0%	0%	0.02%	0.16%
ν = 257%	0.02%	0.04%	0.10%	0.29%

p_+	30%	40%	50%	60%	70%
μ = 4, ν = 257%	-0.11%	0%	0.10%	0.23%	0.40%

⇨ $\mu = 4,\ p_+ = \frac{1}{2}$: relative mismatch $\frac{\widehat{\sigma}_{VS}}{\widehat{\sigma}_{Logswap}} - 1$ is $\approx$ $0.10\%/20\% = 0.5\%$

⇨ Direct backtesting on index returns? Slightly lower estimate:

$$\frac{1}{2}\left(\frac{\langle r^2\rangle}{\langle 2(e^r-1)-2r\rangle} - 1\right) \qquad r \text{ daily log-return}$$

⇨ Conclusion: $\widehat{\sigma}_{VS} - \widehat{\sigma}_{Logswap}$: small impact of 1-day smile

⇨ Mostly impacted by dividend model

Conclusion

- SV model with handle on 1-day smile

  ... while keeping break-even levels of vommas & vannas unchanged

$$r_i \;=\; \sigma_i \sqrt{\delta t}\; z_i$$

  - Fwd variance model: sets scale σ_i of daily returns
  - Additional parameters govern 1-day smile: μ_+, μ_-, p_+
  - Simulation no harder than in std 2F model

- Allows assessment of 1-day smile risk on derivatives
- Unhedgeable risk we're carrying: needs to be priced conservatively

⇨ Near-ATMF smile overwelmingly generated by covariance of spot and ATMF/VS vols

$$\left.\frac{d\widehat{\sigma}_{KT}}{d\ln K}\right|_F \;=\; \frac{1}{2\widehat{\sigma}_T^3 T}\int_0^T \frac{T-t}{T}\left\langle d\ln S_t \; d\widehat{\sigma}_{T,t}^2\right\rangle$$

⇨ 1-day smile impacts tails of vanilla smile – mostly OTM calls

⇨ Larger impact on path-dep payoffs referencing daily returns

  - Daily cliquets
  - Var swaps
  - Capped VSs, absswaps ...